ECON 311 FC Macroeconomic Theory and Policy: Part-I
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Contents

CHAPTER 1

Mathematical Preliminaries

The first Chapter of these notes collects the mathematics needed for the rest of the course. Most of this should be revision, and we'll cover it in class very quickly. You need to know all the material from Sections 1, 2, 3, 4, and 7, and the first part of Section 5.1. You can leave the later parts of Section 5.1 until you start to apply them and see how much you need, and how difficult you find the material. The material in Sections 5.2 and 6 are optional. If you have some taste and ability in this direction it will improve your understanding of optimisation problems, and if you intend to undertake more advanced study in Economics you'll need to learn this material, but it will not be needed for this course.

1. Basic Facts About Differentiation

This section will be quite brief, since you should know this material already. If it proves too brief you might need to review some of your material from Maths 108.

- A Constant:

$$f(x) = a, \qquad f'(x) = 0.$$

- A Linear function:

$$f(x) = x, \qquad f'(x) = 1.$$

- Power Function:

$$f(x) = x^a, \qquad f'(x) = ax^{a-1}.$$

$$f(x) = bx^a, \qquad f'(x) = abx^{a-1}.$$

- Sum rule:

$$f(x) = g(x) + h(x), \qquad f'(x) = g'(x) + h'(x).$$

$$f(x) = ax + cx^b, \qquad f'(x) = a + cbx^{b-1}.$$

- Product rule:

$$f(x) = g(x)h(x), \qquad f'(x) = g'(x)h(x) + g(x)h'(x).$$

- Quotient rule:

$$f(x) = \frac{g(x)}{h(x)}, \qquad f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}.$$

- Chain rule:

$$f(x) = g(y) \text{ where} y = h(x), \qquad f'(x) = g'(y)h'(x).$$

2. Some Common Functions

2.1. Exponentials and Logs. Let us suppose that x is some real number. Then by x^2 we mean $x \times x$, by x^3 we mean $x \times x \times x$, and by x^n where n is some positive whole number we mean $x \times x \times \cdots \times x$ where we multiply x by itself n times.

Thus $x^2 \times x^3 = (x \times x) \times (x \times x \times x)$ which is $x \times x \times x \times x \times x$ or x^5. Similarly, for any two positive whole numbers m and n we have $x^m \times x^n$ equals $x^{(m+n)}$.

We also see that $(x^m)^n$ equals $x^m \times x^m \times \cdots \times x^m$ where we multiply x^m by itself n times, or equivalently, we multiply x by itself $m \times n$ times. Thus $(x^m)^n = x^{(m \times n)}$ (or x^{mn}).

We also use, again supposing n is some positive whole number, the notation $x^{1/n}$ to mean the nth root of x, or, that positive number that when multiplied by itself n times gives x.

We also use the notation x^{-n} to mean $1/(x^n)$. Thus $x^n \times x^{-n} = 1$ and we extend or definition of x^n to allow n to be zero (as long as $x \neq 0$) and use the convention $x^0 = 1$.

Also, since we have defined both x^n and $x^{(1/n)}$ we can extend our definition to define $x^{(a/b)}$ where a is any integer and b is any integer not equal to zero.

Thus we have defined x^z for any value of x and any rational z. (Though we do have to be careful now what we say about negative values of x.) We can now use this definition to define x^z for any positive value of x and any real value of z.

Let us collect the rules for manipulating the exponential functions that we have discussed above so that we shall have them conveniently available for reference.

$$x^a x^b = x^{(a+b)} \tag{1}$$

$$(x^a)^b = x^{ab} \tag{2}$$

$$x^{(1/a)} = \sqrt[a]{x} \tag{3}$$

$$x^{-a} = 1/x^a \tag{4}$$

EXERCISE 1.1. Use the above rules to find expressions for

$$x^a/x^b, \text{ and}$$

$$(x^a)^{(1/b)}.$$

We have implicitly been thinking of x as the variable and treating z (or n) as a parameter. If we think of x as a parameter and think of x^z as a function of z we call it an exponential function.

One particular important possible value of x is the constant e (equal approximately to 2.712. The function e^z of the variable z is often called *the* exponential function. (We'll see in a little bit one reason it is so important.) Other important cases of exponential functions are the functions 10^z and 2^z, the latter particularly important in information theory and computer science.

The inverse of the exponential function is called the log or logarithm function. Having defined the exponential function we can define the log function by letting $\log y$ be that value z such that $y = e^z$. When we use the term log we shall always mean log to the base e. If we want to refer to the log to another base we shall say explicitly, for example $\log_{10}$ or $\log_2$. (Another common convention is to use log to mean $\log_{10}$ and ln to mean log (to the base e).

We have similar rules for operations with logs as with the exponential function.Thus

$$\log(ab) = \log a + \log b \tag{5}$$

$$\log(a^b) = b \log a \tag{6}$$

$$\log(1/a) = -\log a. \tag{7}$$

2.1.1. *Differentiating log and exponential functions.* One reason for choosing the value of e as the base for the standard exponential and log functions is that it makes differentiation (and integration) very convenient. We state here without proof, or further discussion the following two facts.

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}.$$

EXERCISE 1.2. What is $\frac{\mathrm{d}}{\mathrm{d}x}10^x$? What is $\frac{\mathrm{d}}{\mathrm{d}x}\log_{10} x$? [Hint: use the rules we discussed above to express 10^x as a constant times e^x. It would be cleaner to express your answer in terms of 10^x rather than in terms of e^x. Do something similar for $\log_{10} x$.]

3. Taylor Series Approximation

The derivative of a function may be thought of as providing a *linear approximation* to the function. That is, given a function[1] $f : \mathbb{R} \to \mathbb{R}$ and the derivative of f at some point x^0 we have

$$f(x) \approx f(x^0) + f'(x^0)(x - x^0), \tag{8}$$

that is, near x^0 we can find $f(x)$ approximately as the sum of $f(x^0)$ and some constant times the difference $x - x^0$. This constant is precisely the derivative of the function f at x^0. The Taylor series gives us more precise information about this approximation. We shall only deal with the *second order* Taylor series expansion.

$$f(x) = f(x^0) + f'(x^0)(x - x^0) + f''(t)(x - x^0)^2/2, \tag{9}$$

for some t between x^0 and x. Notice that for values of x close to x^0 the quantity $(x - x^0)^2$ will be very small (and positive), being the square of a small number, and $f''(t)$ will be very close to $f''(x^0)$.

We should be a little more careful about what we say. There are functions that are not differentiable. There are functions that are differentiable, but for which the derivative is not a contu=inuous function. There are functions that are continuously differentiable (that is, for which the derivative exists and is continuous) but for which the second derivative does not exist. What we need here is that the first and second derivatives exist and are continuous. We simply assume that our function satisfies these conditions, that is, what we have stated here is true for functions that are *twice continuously differentiable.*

4. Unconstrained Maximisation

Suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$ (that is twice continuously differentiable) and we want to know if the function f attains its maximum at some point x^0. From the previous section we know that, for any other point x we have

$$f(x) = f(x^0) + f'(x^0)(x - x^0) + f''(t)(x - x^0)^2/2,$$

for some t between x^0 and x.

Now, if x is close to x^0 then $|x - x^0|$ is much bigger than $(x - x^0)^2$. Thus if $f'(x^0) > 0$ we could pick some point x just a little bigger than x^0 and the second term would be positive and bigger than the third term, and so $f(x)$ would be bigger than $f(x^0)$. Thus the function could not be at a maximum at x^0. Similarly, if $f'(x^0) < 0$ we could pick some point x just a little smaller than x^0 and the second term would be positive and bigger than the third term, and so $f(x)$ would be bigger than $f(x^0)$. Again, the function could not be at a maximum at x^0. Thus if the function is at a maximum at x^0 then it must be that $f'(x^0) = 0$.

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[1] This means a function that specifies for any real number x some real number $f(x)$. The symbol $\mathbb{R}$ means the real numbers.

Now, if $f'(x^0)=0$ the Taylor series becomes

$$f(x)=f(x^0)+f''(t)(x-x^0)^2/2,$$

for some t between x^0 and x. Now, if $f''(x^0)>0$, then we could pick some point $x\neq x^0$ and close enough to x^0 so that $f''(t)>0$ for all values of t between x^0 and x. Notice that $(x-x^0)^2>0$. Thus $f''(t)(x-x^0)^2>0$ and so $f(x)>f(x^0)$, and the function could not be at a maximum at x^0.

Thus if f attains its maximum at x^0 it must be that $f'(x^0)=0$ and $f''(t)(x-x^0)^2\leq 0$. These are called the first and second order necessary conditions for a maximum.

If $f'(x^0)=0$ and $f''(x^0)<0$ then, at least for values of x close enough to x^0 that $f''(t)<0$ for all values of t between x^0 and x we have that $f(x)\leq f(x^0)$ and so the function attains at least a local maximum at x^0. These conditions are called the first and second order sufficient conditions for a local maximum.

Suppose that we have more than one dimension, that is that our function is $f:\mathbb{R}^K\to\mathbb{R}$. It is possible to develop systematically the first and second necessary and sufficient conditions exactly analogous to the conditions we found for the one dimensional problem. We won't be quite so systematic and will not say anything about the second order conditions.

It is, however, very easy to see what the first order necessary conditions should be. If the function f attains its maximum at $x^0\in\mathbb{R}^K$, then the function $f(\cdot,x_2^0,\dots,x_K^0):\mathbb{R}\to\mathbb{R})$ is exactly the kind of function we considered above, and it attains its maximum at x_1^0. Thus (one of) the first order necessary conditions is that $\partial f/\partial x_1=0$. Moreover, we can make this argument, not just for the first dimension, that is for the variable x_1, but for any of the dimensions. Thus the first order necessary conditions are

$$\frac{\partial f}{\partial x_k}(x^0)=0,\text{ for }k=1,\dots,K. \tag{10}$$

5. Constrained Maximisation

5.1. Lagrange Multipliers. Consider the problem of a consumer who seeks to distribute his income across the purchase of the two goods that he consumes, subject to the constraint that he spends no more than his total income. Let us denote the amount of the first good that he buys x_1 and the amount of the second good x_2, the prices of the two goods p_1 and p_2, and the consumer's income y. The utility that the consumer obtains from consuming x_1 units of good 1 and x_2 of good two is denoted $u(x_1,x_2)$. Thus the consumer's problem is to maximise $u(x_1,x_2)$ subject to the constraint that $p_1x_1+p_2x_2\leq y$. (We shall soon write $p_1x_1+p_2x_2=y$, i.e., we shall assume that the consumer must spend all of his income.) Before discussing the solution of this problem lets write it in a more "mathematical" way.

$$\begin{aligned}&\max_{x_1,x_2} && u(x_1,x_2)\\ &\text{subject to } p_1x_1+p_2x_2=y\end{aligned} \tag{11}$$

We read this "Choose x_1 and x_2 to maximise $u(x_1,x_2)$ subject to the constraint that $p_1x_1+p_2x_2=y$."

Let us assume, as usual, that the indifference curves (i.e., the sets of points (x_1,x_2) for which $u(x_1,x_2)$ is a constant) are convex to the origin. Let us also assume that the indifference curves are nice and smooth. Then the point (x_1^*,x_2^*) that solves the maximisation problem (11) is the point at which the indifference curve is tangent to the budget line as given in Figure 1.

One thing we can say about the solution is that at the point (x_1^*,x_2^*) it must be true that the marginal utility with respect to good 1 divided by the price of good 1 must equal the marginal utility with respect to good 2 divided by the price of good 2. For if this were not true then the consumer could, by decreasing the consumption of the good for which this ratio was lower and increasing the consumption of the other good, increase his utility.

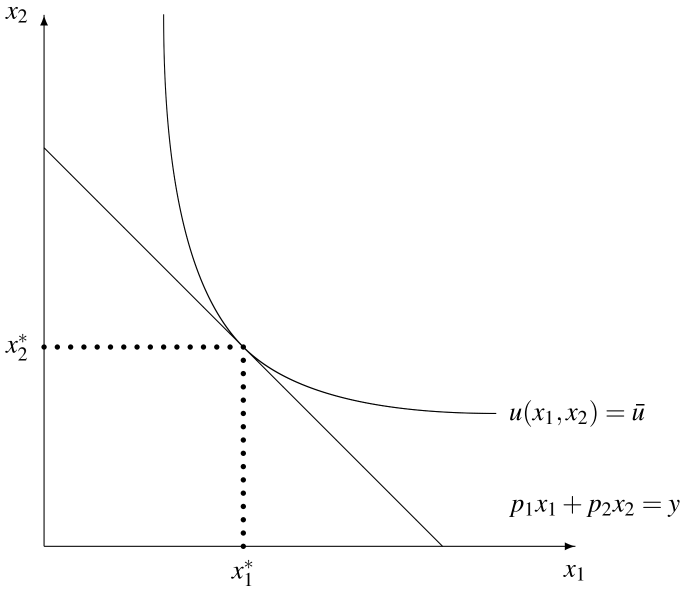

FIGURE 1. The Utility Maximisation Problem

Marginal utilities are, of course, just the partial derivatives of the utility function. Thus we have

$$\frac{\frac{\partial u}{\partial x_1}(x_1^*,x_2^*)}{p_1}=\frac{\frac{\partial u}{\partial x_2}(x_1^*,x_2^*)}{p_2}. \tag{12}$$

The argument we have just made seems very “economic.” It is easy to give an alternate argument that does not explicitly refer to the economic intuition. Let x_2^u be the function that defines the indifference curve through the point (x_1^*,x_2^*), i.e.,

$$u(x_1,x_2^u(x_1))\equiv\bar{u}\equiv u(x_1^*,x_2^*).$$

Now, totally differentiating this identity gives

$$\frac{\partial u}{\partial x_1}(x_1,x_2^u(x_1))+\frac{\partial u}{\partial x_2}(x_1,x_2^u(x_1))\frac{dx_2^u}{dx_1}(x_1)=0.$$

That is,

$$\frac{dx_2^u}{dx_1}(x_1)=-\frac{\frac{\partial u}{\partial x_1}(x_1,x_2^u(x_1))}{\frac{\partial u}{\partial x_2}(x_1,x_2^u(x_1))}.$$

Now $x_2^u(x_1^*)=x_2^*$. Thus the slope of the indifference curve at the point (x_1^*,x_2^*)

$$\frac{dx_2^u}{dx_1}(x_1^*)=-\frac{\frac{\partial u}{\partial x_1}(x_1^*,x_2^*)}{\frac{\partial u}{\partial x_2}(x_1^*,x_2^*)}.$$

Also, the slope of the budget line is $-\frac{p_1}{p_2}$. Combining these two results again gives result (12).

Since we also have another equation that (x_1^*,x_2^*) must satisfy, viz

$$p_1x_1^*+p_2x_2^*=y \tag{13}$$

we have two equations in two unknowns and we can (if we know what the utility function is and what p_1, p_2, and y are) go happily away and solve the problem. (This isn’t quite true but we shall not go into that at this point.) What we shall develop is a systemic and useful way to obtain the conditions (12) and (13). Let us first denote the common value of the ratios in (12) by λ. That is,

$$\frac{\frac{\partial u}{\partial x_1}(x_1^*,x_2^*)}{p_1} = \lambda = \frac{\frac{\partial u}{\partial x_2}(x_1^*,x_2^*)}{p_2}$$

and we can rewrite this and (13) as

$$\begin{aligned}\frac{\partial u}{\partial x_1}(x_1^*,x_2^*) - \lambda p_1 &= 0\\ \frac{\partial u}{\partial x_2}(x_1^*,x_2^*) - \lambda p_2 &= 0\\ y - p_1x_1^* - p_2x_2^* &= 0.\end{aligned} \tag{14}$$

Now we have three equations in x_1^*, x_2^*, and the new artificial or auxiliary variable λ. Again we can, perhaps, solve these equations for x_1^*, x_2^*, and λ. Consider the following function

$$\mathscr{L}(x_1,x_2,\lambda) = u(x_1,x_2) + \lambda(y - p_1x_1 - p_2x_2) \tag{15}$$

This function is known as the Lagrangian. Now, if we calculate $\frac{\partial \mathscr{L}}{\partial x_1}$, $\frac{\partial \mathscr{L}}{\partial x_2}$, and, $\frac{\partial \mathscr{L}}{\partial \lambda}$, and set the results equal to zero we obtain exactly the equations given in (14). We now describe this technique in a somewhat more general way.

Suppose that we have the following maximisation problem

$$\begin{aligned}&\max_{x_1,\dots,x_n} \quad f(x_1,\dots,x_n)\\ &\text{subject to } g(x_1,\dots,x_n) = c\end{aligned} \tag{16}$$

and we let

$$\mathscr{L}(x_1,\dots,x_n,\lambda) = f(x_1,\dots,x_n) + \lambda(c - g(x_1,\dots,x_n)) \tag{17}$$

then if $(x_1^*,\dots,x_n^*)$ solves (16) there is a value of λ, say λ^* such that

$$\frac{\partial \mathscr{L}}{\partial x_i}(x_1^*,\dots,x_n^*,\lambda^*) = 0 \qquad i = 1,\dots,n \tag{18}$$

$$\frac{\partial \mathscr{L}}{\partial \lambda}(x_1^*,\dots,x_n^*,\lambda^*) = 0. \tag{19}$$

Notice that the conditions (18) are precisely the first order conditions for choosing $x_1,\dots,x_n$ to maximise $\mathscr{L}$, once λ^* has been chosen. This provides an intuition into this method of solving the constrained maximisation problem. In the constrained problem we have told the decision maker that he must satisfy $g(x_1,\dots,x_n) = c$ and that he should choose among all points that satisfy this constraint the point at which $f(x_1,\dots,x_n)$ is greatest. We arrive at the same answer if we tell the decision maker to choose any point he wishes but that for each unit by which he violates the constraint $g(x_1,\dots,x_n) = c$ we shall take away λ units from his payoff. Of course we must be careful to choose λ to be the correct value. If we choose λ too small the decision maker may choose to violate his constraint—e.g., if we made the penalty for spending more than the consumer's income very small the consumer would choose to consume more goods than he could afford and to pay the penalty in utility terms. On the other hand if we choose λ too large the decision maker may violate his constraint in the other direction, e.g., the consumer would choose not to spend any of his income and just receive λ units of utility for each unit of his income.

It is possible to give a more general statement of this technique, allowing for multiple constraints. (Of course, we should always have fewer constraints than we have variables.)

Suppose we have more than one constraint. Consider the problem

$$\begin{aligned}\max_{x_1,\dots,x_n} \quad & f(x_1,\dots,x_n)\\ \text{subject to } g_1(x_1,\dots,x_n) &= c_1\\ \vdots \qquad & \quad \vdots\\ g_m(x_1,\dots,x_n) &= c_m.\end{aligned}$$

Again we construct the Lagrangian

$$\begin{aligned}\mathscr{L}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) &= f(x_1,\dots,x_n)\\ &+\lambda_1(c_1-g_1(x_1,\dots,x_n))+\cdots+\lambda_m(c_m-g_m(x_1,\dots,x_n))\end{aligned} \tag{20}$$

and again if $(x_1^*,\dots,x_n^*)$ solves (20) there are values of λ, say $\lambda_1^*,\dots,\lambda_m^*$ such that

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial x_i}(x_1^*,\dots,x_n^*,\lambda_1^*,\dots,\lambda_m^*) &= 0 \qquad & i=1,\dots,n\\ \frac{\partial\mathscr{L}}{\partial \lambda_j}(x_1^*,\dots,x_n^*,\lambda_1^*,\dots,\lambda_m^*) &= 0 \qquad & j=1,\dots,m.\end{aligned} \tag{21}$$

6. Expectation and Variance

6.1. Definition. Here we wish to record some useful facts about random variables. We shall not give a precise definition of a random variable. The basic idea is that a random variable may take on a number of different values and to each value we associate some probability.[2]

Let us first consider the case that the random variable X takes on only a finite number of values, $x^1,\dots,x^K$, with each x^k some real number, and that the probability that $X=x^k$ is p^k. The *expectation* of X is defined to be

$$E(X)=\sum_{k=1}^{K}p^kx^k. \tag{22}$$

If the random variable X could take on any real value between a and b and had *density function* f such that

$$\Pr\{c\le X\le d\}=\int_c^d f(x)\mathrm{d}x \tag{23}$$

then we would define the expectation of the random variable to be

$$E(X)=\int_a^b xf(x)\mathrm{d}x. \tag{24}$$

More generally, if X has a *distribution function* F such that

$$\Pr\{X\le c\}=\int_a^c f(x)\mathrm{d}x \tag{25}$$

then

$$E(X)=\int_a^b x\mathrm{d}F(x), \tag{26}$$

where the last is the *Stieltjes integral*.

The expected value os a random variable is also called the *mean* of the random variable. Notice that $E(X)$ is not a random variable. While the random variable X may take on a number of different values $E(X)$ is the (weighted) average of those values. The expected

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[2] To be more accurate, and to include the case in which the random variable may take on, for example, any positive value, we should say that probability is associated to any set of values. Thus if the random variable could take on any value in the interval $[0,1]$, we should say what is the probability of any interval $[a,b]$ with $0\le a\le b\le 1$.

value of the random variable is one of the measures of central tendency of the random variable.[3] As well as a measure of central tendency we are often interested in how *variable* or *disperse* the random variable is, that is how far away it is from its expected value, on average. The most common measure of dispersion is the *variance*. The variance is defined to be the expected value of the square of the difference between the random variable and its mean. That is,

$$Var(X) = E((X - E(X))^2). \tag{27}$$

For two random variables X and Y we are often interested in whether the random variables tend to move together so that large values of X tend to occur with large values of Y or not. One measure of this is the covariance. The covariance of X and Y is defined as the expected value of the product of the differences between the random variable and its mean, that is,

$$Cov(X,Y) = E((X - E(X))(Y - E(Y))). \tag{28}$$

6.2. Rules of Expectation and Variance. Here we give some useful rules regarding expectation and variance.

- A constant a (that is, a random variable that takes on only one value, a):

$$\begin{aligned} E(a) &= a, \\ Var(a) &= 0. \end{aligned}$$

- Product of a constant b and a random variable X:

$$\begin{aligned} E(bX) &= bE(X), \\ Var(bX) &= b^2 Var(X). \end{aligned}$$

- Sum and product:

$$\begin{aligned} E(a + bX) &= a + bE(X), \\ Var(a + bX) &= b^2 Var(X). \end{aligned}$$

$$\begin{aligned} E(aX + bY) &= aE(X) + bE(Y), \\ Var(aX + bY) &= a^2 Var(X) + b^2 Var(Y) + 2abCov(X,Y). \end{aligned}$$

- Expectation of the product of two random variables:

$$\begin{aligned} E(XY) &= E(X)E(Y) + Cov(X,Y), \\ E\left(X^2\right) &= E(X)^2 + Var(X). \end{aligned}$$

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[3] Other measures of central tendency are the median and the mode.

CHAPTER 2

Output, Inputs, and Growth

Read Auerbach and Kotlikoff, Chapter 1. Concentrate on the definitions and the basic analytics. Don't worry about the case studies (unless you are interested). We shall concentrate on developing the model in the first half of the course. Dr. Dominguez will tell you more about some of the facts about the macroeconomy in the second half of the course.

1. Growth rates and average growth rates

Suppose that the economy grows by 4% this year and 0% next year. What is the average growth rate over the two years? Suppose that the economy of New Zealand grows by 4% this year and the economy of the United States grows by 0% this year. What is the average growth rate for the two countries?

There are a number of possible answers to these questions. One is to say: I know what an average is. I'll just add 4 and 0, get 4, divide by 2, and the average is 2. This is not *incorrect* but it's not very useful either. Rather what we usually mean by "average" in cases like this is, what common value in the two cases would lead to the same overall effect. For the first question we ask: Suppose the growth rate was g in both years. For what value of g would we reach the same level as when the growth rate was 4% this year and 0% next year?

The answer to the second question is similar, though we shall see that it results in a quite different formula. We ask: Suppose that New Zealand and the United States had grown at the same rate, g. For what value of g would the total size of both economies have been the same as if New Zealand had grown by 4% and the United States grown by 0%?

EXERCISE 2.1. Find the relevant averages in the two cases described above.

Usually when I give exercises I'll leave them for you to solve. I hope that you've at least given the problem some thought, before reading this, but I'll give the answer now.

Suppose the total size of the economy at the beginning of this year is X. Then, if the economy grows at 4% this year, at the beginning of next year the size of the economy will be $1.04X$ and at the beginning of the following year it will again be $1.04X$, since it grows by 0% in that year. On the other hand, if it had grown at a rate g in each year, at the beginning of next year the size of the economy will be $(1+g)X$ and at the beginning of the following year it will again be $(1+g)(1+g)X = (1+g)^2X$. Thus we have to find the value of g that $(1+g)^2 = 1.04$ or $1+g = \sqrt{1.04} = 1.0198$ or $g = 0.0198 = 1.98\%$. If we call the the term $1+g$ the growth *factor* then we can describe the process of finding the average as finding the *geometric* average of the growth factors.

This is how we shall typically be using growth rates in this course. However, as the second question indicates this is not the only way in which an average of growth rates might arise, and this is not the only way that we might take an average of growth rates. Suppose that the economy of New Zealand was of size X at the beginning of the year and the economy of the United States was of size Y. Then the average growth rate that would answer our question would be $(X \times 4\% + Y \times 0\%)/(X+Y)$. That is, the weighted arithmetic average of the growth rates.

EXERCISE 2.2. Can you think of any questions for which the simple arithmetic average of the growth rates would be the right answer?

Another "growth" rate that we are interested in is the *interest* rate, which we can think of as the rate at which the money in our savings account would grow, if we do not add to the principal. If the interest rate is r and at the beginning of the year your savings are S then at the beginning of next year you will have $(1+r)S$ and at the beginning of the following year, $(1+r)^2S$.

2. The Cobb-Douglas Production Function

In macroeconomics we are typically interested in aggregate behaviour. Thus we shall ignore differences in capital goods, types of labour, and various goods produced in the economy. We assume that there is, in each period, a certain amount of capital, which we shall denote K, a certain amount of labour, which we shall denote L, and that these are combined in a production process to give a certain amount of output, which we shall denote Y. (When we come to discuss the behaviour of the economy over time we shall index each of these by a subscript denoting the time period.) We assume that once we know the amounts of labour and capital that tells you the amount of output, that is Y *is a function of* K and L. Let us denote this *production function* F and write

$$Y = F(K,L).$$

Let us first look at how the level of output changes as we change the scale of production. Informally, when we double the inputs we might double the level of output, more than double the level, or less than double the level. We shall describe these cases as *constant returns to scale*, *increasing returns to scale*, and *decreasing returns to scale*.

More formally, if for any levels of capital and labour, K and L, and any positive constant $\lambda > 0$, we have $F(\lambda K, \lambda L) = \lambda F(K,L)$ we say the production exhibits *constant returns to scale*. If for any levels of capital and labour, K and L, and any positive constant $\lambda > 1$, we have $F(\lambda K, \lambda L) > \lambda F(K,L)$ we say the production exhibits *increasing returns to scale*. If for any levels of capital and labour, K and L, and any positive constant $\lambda > 1$, we have $F(\lambda K, \lambda L) < \lambda F(K,L)$ we say the production exhibits *decreasing returns to scale*. Notice that we only say a production function exhibits increasing returns to scale if it exhibits such behaviour everywhere, that is for all values of K and L, and any $\lambda > 1$. Of course some functions may not exhibit the same properties everywhere. They may seem to have increasing returns close to some combination of K and L and decreasing returns at other values.

For some purposes this level of generality might be enough. For other purposes, and everywhere in at least the first half of this course, it is convenient to be a little more explicit. Thus, we assume that the production function F has the Cobb-Douglas form, that is, it is given by

$$Y = AK^{\beta}L^{1-\beta}. \tag{29}$$

The coefficient A is called the level of multifactor productivity. We shall see that in competitive markets the owners of capital and labour will receive shares, β and $1-\beta$ of the output Y. In general, to the extent that the Cobb-Douglas production is a good approximation to reality the parameter β seems to be quite stable, and to be around .30 in magnitude.

We have here a constant returns to scale Cobb-Douglas production function. Had we chosen exponents on K and L that summed to less than 1 we would have a decreasing returns to scale production function; had we chosen exponents on K and L that summed to more than 1 we would have an increasing returns to scale production function.

2.1. Accounting for growth.

Let is now consider two periods, period t and period $t+1$, and let us assume that each of the variables K, L, and Y, and the parameter A can take on different values in the two periods.

Thus

$$Y_t = A_t K_t^{\beta} L_t^{1-\beta} \tag{30}$$

and

$$Y_{t+1} = A_{t+1} K_{t+1}^{\beta} L_{t+1}^{1-\beta}. \tag{31}$$

If we take the log of each of these equations we obtain

$$\log Y_t = \log A_t + \beta \log K_t + (1-\beta) \log L_t \tag{32}$$

and

$$\log Y_{t+1} = \log A_{t+1} + \beta \log K_{t+1} + (1-\beta) \log L_{t+1}. \tag{33}$$

And now we subtract equation (32) from equation (33) to obtain

$$\log Y_{t+1} - \log Y_t = \log A_{t+1} - \log A_t + \beta(\log K_{t+1} - \log K_t) + (1-\beta)(\log L_{t+1} - \log L_t). \tag{34}$$

For any variable X, as long as the percentage change in X is small the difference $\log X_{t+1} - \log X_t$ is very close to the percentage change in X, that is to $(X_{t+1} - X_t)/X_t = \Delta X/X_t$, and since when we use the Δ notation for the change we do not need to specify the time period we shall write the last term simply as $\Delta X/X$. Thus we can re-write equation (34) as

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \beta \frac{\Delta K}{K} + (1-\beta)\frac{\Delta L}{L}. \tag{35}$$

Equation (35) says that we can decompose a percentage change in output, into the percentage change in multifactor productivity plus β times the percentage change in the amount of capital used plus $1-\beta$ times the percentage change in the amount of labour used.

2.2. The Cobb-Douglas production function and competitive markets. In a competitive market the factors capital and labour will be paid their marginal product. Thus each unit of labour (worker) will be paid

$$\frac{\partial}{\partial L} Y = \frac{\partial}{\partial L} A K^{\beta} L^{1-\beta} = A K^{\beta} (1-\beta) L^{-\beta} = (1-\beta) A K^{\beta} L^{-\beta} \tag{36}$$

and, in total, labour will be paid this amount times the total amount of labour, L, or

$$A K^{\beta} (1-\beta) L^{-\beta} L = (1-\beta) A K^{\beta} L^{1-\beta} = (1-\beta) Y. \tag{37}$$

Thus, in total, the owners of labour, the workers, obtain a fraction $1-\beta$ of the output. Similarly, the owners of capital obtain a fraction β of the output. Because of our assumption of constant returns to scale the owners of the firms obtain no (supernormal) profits.

EXERCISE 2.3. Explicitly calculate the marginal product of capital and the total amount of output obtained by the owners of capital in a competitive market, as we did for the owners of labour, above.

3. Labour Productivity

We now consider the ration Y/L the amount of output per unit of labour input. This ratio is called *labour productivity*. If we divide both sides of equation (29) by L we obtain

$$\frac{Y}{L} = \frac{A K^{\beta} L^{1-\beta}}{L} = A \left(\frac{K}{L}\right)^{\beta}, \tag{38}$$

which we call the *Cobb-Douglas production function in intensive form*.

A fairly common convention is to use lower case letters to refer to pre capita versions of the variables on the model. Thus we write y for the ratio Y/L and k for the ratio K/L. (We use the standard symbol, 1, for the ratio L/L.[1]) Thus we can rewrite equation (38) as

$$y = Ak^{\beta}. \tag{39}$$

Recall that we saw that each worker will be paid his marginal product and we found in equation (36) that this would be

$$(1-\beta)AK^{\beta}L^{-\beta} = (1-\beta)Ak^{\beta} = (1-\beta)y. \tag{40}$$

Thus, at least for the Cobb-Douglas production function, the marginal product of labour, which in a competitive labour market will be the same as the wage rate, has the very simple form of $1-\beta$ times labour productivity.

4. Capital Productivity

Just as we considered the marginal product of labour we may also consider the marginal product of capital. In competitive capital markets the owners of capital will be paid the marginal product of capital for each unit of capital they provide. Thus each unit of capital will be paid

$$\frac{\partial}{\partial K}Y = \frac{\partial}{\partial K}AK^{\beta}L^{1-\beta} = A\beta K^{\beta-1}L^{1-\beta} = \beta A\left(\frac{K}{L}\right)^{\beta-1} = \beta Ak^{\beta-1} \tag{41}$$

and, in total, capital will be paid this amount times the total amount of capital, K, or

$$A\beta K^{\beta-1}L^{1-\beta}K = \beta AK^{\beta}L^{1-\beta} = \beta Y. \tag{42}$$

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[1] This is what is called in advanced textbooks, a joke, though perhaps not a very good one.

CHAPTER 3

The Basic Overlapping Generations Model of the Economy

Read Auerbach and Kotlikoff, Chapters 2 and 3. A more advanced (and shorter) treatment may be found in Chapter 3 of Blanchard and Fischer (1990).

1. Overview

The model we consider in this Chapter is one of the basic ones used in modern macroeconomics. The idea of using such a framework was introduced by Maurice Allais[1] (1947) and Paul Samuelson[2] (1958). It was developed in more or less the form we shall give by Peter Diamond (1965). This model forms an important part of Kydland and Prescott's 1988 paper called "Time to Build and Aggregated Fluctuations," one of the seminal works in the *real business cycle* theories.[3]

The basic idea of the model is that in each period a new generation of economic agents is born. Each individual lives for two periods, inelastically[4] supplying one unit of labour in his first period of life and consuming in both periods of his life.

Each agent from generation t supplies one unit of labour to the production process at the beginning of his first period of life in order to produce period t's output of the single good of this economy. For this he receives whatever the wage rate is in period t. Part of this he consumes and part he saves. The part he saves becomes the capital used in the production process in period $t+1$. He provides the part of his wage that he didn't consume in period t at the beginning of period $t+1$ in return for a greater amount of the good for him to consume at the end of period $t+1$. The extra bit that they get we call the interest rate for period $t+1$.

2. The Consumption-Saving Decision

2.1. The intertemporal budget constraint. Recall from your microeconomics the idea of a budget constraint. This is the set of consumption bundles that are affordable for the consumer. This set typically depends on the consumer's wealth (sometimes we say income, but wealth is more accurate) and the prices of the various consumption goods. A similar situation arises here.

Though it isn't really very complicated we'll develop the intertemporal budget constraint a step at a time. Let us consider the generation born in period t. We have said that when the individuals in this generation are young the supply one unit of labour for which they receive the wage of that period, w_t. This amount they can either consume in period t or take over into period $t+1$ as assets. We'll call the consumption in period t of such an individual c_{yt} (y for young and t for the time period) and the amount of assets this individual takes over into period $t+1$ we'll call a_{t+1}. Both c_{yt} and a_{t+1} are measured in units of

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[1] Allais received the Nobel Prize in Economics in 1988

[2] Samuelson received the Nobel Prize in Economics in 1970

[3] Kydland and Prescott received the Nobel Prize in Economics in 2004

[4] This means that his supply of labour is fixed; it does not depend on the price system, and in particular does not depend on the price of labour, that is, the wage rate.

the one good produced in period t. We'll assume that the wages are also measured in units of this good so we have

$$a_{t+1} = w_t - c_{yt}. \tag{43}$$

In period $t+1$ this individual will be old and will not supply any labour. He will supply his assets, a_{t+1} to the production process and will receive, in return, at the end of the period $a_{t+1}(1+r_{t+1})$ which he will consume. Thus there is no depreciation of capital in this model. For every unit of the (capital) good individual supplies the production process, he is paid r_{t+1}, and also receives back the full unit of the good after the production process is completed. Sadly, this individual will then die. We'll call the amount he consumes in period $t+1$ as an old person c_{ot+1} (o for old and $t+1$ for the period in which he consumes it). Again, we can write this as a budget constraint, namely

$$c_{ot+1} = a_{t+1}(1+r_{t+1}). \tag{44}$$

Of course, what the consumer actually cares about is the amount he consumes is each period. He doesn't directly care about the amount of assets he takes over from period t to period $1+1$. If we combine equations (43) and (44) by substituting equation (43)'s formula for a_{t+1} into equation (44) we obtain a budget constraint that relates the two quantities that the individual does directly care about, his consumption in each period, namely

$$c_{yt} + \frac{c_{ot+1}}{1+r_{t+1}} = w_t. \tag{45}$$

The term $1/(1+r_{t+1})$ is the price of period $t+1$ consumption in terms of period t consumption, it is the number of units of period t consumption that the individual would have to give up in order to obtain an additional unit of period $t+1$ consumption, just as a price of a good in dollars represents the number of dollars one would give up in order to obtain an additional unit of the good.

2.2. Intertemporal Preferences. So we have one half of the information we need to determine how the individuals of this generation will behave. We have said what choices they have available to them. We also need to say what they like or what they want, that is, we have to describe their preferences. As usual we do function this by specifying a utility function. We shall assume that their utility has the same Cobb-Douglas form as the production function.[5] In particular we assume that the individuals of generation t have the utility function

$$u_t(c_{yt}, c_{ot+1}) = c_{yt}^{\alpha} c_{ot+1}^{1-\alpha}, \tag{46}$$

with $0 < \alpha < 1$. Notice that we have labelled the utility function with the subscript t to indicate that it is generation t preferences we are describing. However we assume that each generation has the same preferences. The individuals of generation t may have a different level of utility than another generation, but only because their levels of consumption may differ. We assume that the functional relationship describing how the level of the individual's utility depends on c_{yt} and c_{ot+1} is the same for each generation, that is we assume that each generation has the same intertemporal preferences.

The parameter α represents the consumer's *time preference*. If α equals 1 then the consumer cares only about his consumption when he is young and completely discounts the future. If α equals a half he cares equally about consumption in each period.

When dealing with intertemporal problems we often assume that an individual's preferences are described by a utility function that is the discounted sum of "period" utility utility function. In this two period case this would mean we would describe the individual's utility as

$$u_t(c_{yt}, c_{ot+1}) = v(c_{yt}) + \delta v(c_{ot+1}), \tag{47}$$

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[5] We make this assumption on the form of the utility function for this chapter. Later in the course you will see the more general "constant elasticity of substitution" or CES utility function.

and we would call δ the discount factor and typically assume that $\delta < 1$. While the utility function we have specified above seems quite different form this form, in fact, it represents exactly the same behaviour as a particular utility function of this form.

If we take the log of the function u_t that we defined in equation(46) and assumed that the individual maximised that function instead of u_t we would get exactly the same behaviour. Since the log function is strictly increasing we would choose the same bundle (c_{yt}, c_{ot+1}) to maximise $\log u_t$ as we would to maximise u_t. So suppose instead of considering the utility function u_t we consider the "new" utility function $\tilde{u}_t = \log u_t$. (In fact, just so everything looks nice in the end, lets also multiple by $1/\alpha$ as well. Since we have already assumed that $\alpha > 0$ this won't change the behaviour either.) That is,

$$\begin{aligned} \tilde{u}_t(c_{yt}, c_{ot+1}) &= \frac{1}{\alpha} \log u_t(c_{yt}, c_{ot+1}) \\ &= \frac{1}{\alpha} \alpha \log c_{yt} + \frac{1}{\alpha}(1-\alpha) \log c_{ot+1} \\ &= \log c_{yt} + \tfrac{1-\alpha}{\alpha} \log c_{ot+1}. \end{aligned} \tag{48}$$

We observe that this utility function does indeed have the form of a discounted sum of each period's utility (in this case the per period utility function is simply the log of that period's consumption) and that the discount factor is $(1-\alpha)/\alpha$, which is zero when α equals 1, and equals 1 when α equals a half, which is what we would expect from what we saw about how the individual's time preference depended on α.

2.3. The solution to the intertemporal decision problem. We can state the decision problem of an individual in the generation born in period t, as the following constrained maximisation problem:

$$\begin{aligned} &\max_{c_{yt}, c_{ot+1}} c_{yt}^{\alpha} c_{ot+1}^{1-\alpha} \\ &\text{subject to } c_{yt} + \frac{c_{ot+1}}{1+r_{t+1}} = w_t. \end{aligned} \tag{49}$$

In the next subsection we shall examine in detail how to solve this maximisation problem. Here we simply note a general useful fact about the Cobb-Douglas utility function, namely that in a standard utility maximisation subject to a linear budget constraint problem if the utility function is Cobb-Douglas, then, whatever the relative prices, the consumer will spend a fraction of his income on a particular good equal to the exponent on that good in the Cobb-Douglas utility function.[6] In our case this means that the individual will spend a fraction α of w_t on his consumption when young, that is, since w_t and c_{yt} are measured in the same units (or equivalently the price of c_{yt} is 1)

$$c_{yt} = \alpha w_t. \tag{50}$$

Similarly he will spend the fraction $(1-\alpha)$ on c_{ot+1}. Now, if we look at the budget constraint we see that the price of c_{ot+1} is not 1, but $1/(1+r_{t+1})$. Thus

$$\frac{c_{ot+1}}{1+r_{t+1}} = (1-\alpha) w_t$$

or

$$c_{ot+1} = (1-\alpha)(1+r_{t+1}) w_t. \tag{51}$$

We can also think what this means in terms of the two step constraints we originally used to describe the individual's maximisation problem. In the first period the individual supplies 1 unit of labour and receives w_t. Of this he consumes in that period $c_{yt} = \alpha w_t$, and saves the rest, that amount becoming the assets he takes into period $t+1$,

$$a_{t+1} = (1-\alpha) w_t. \tag{52}$$

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[6] Actually, more generally, the fraction is the exponent divided by the sum of the exponents, but we always make the sum of the exponents equal to 1.

He provides this amount to the production process in period $t+1$ and receives in return $(1+r_{t+1})a_{t+1} = (1-\alpha)(1+r_{t+1})w_t$, which he consumes at the end of period $t+1$.

2.4. A detailed look at the maximisation problem. Here we shall look in detail at how to solve the maximisation problem. You may think of this as an application of the methods described in Section 1 of Chapter 1.

Recall that the problem is

$$\max_{c_{yt},c_{ot+1}} c_{yt}^{\alpha} c_{ot+1}^{1-\alpha}$$

$$\text{subject to } c_{yt} + \frac{c_{ot+1}}{1+r_{t+1}} = w_t.$$

The Lagrangian function is

$$\mathscr{L}(c_{yt}, c_{ot+1}, \lambda) = c_{yt}^{\alpha} c_{ot+1}^{1-\alpha} + \lambda \left(w_t - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1}} \right). \tag{53}$$

and the first order conditions are

$$\frac{\partial \mathscr{L}}{\partial c_{yt}}(c_{yt}, c_{ot+1}, \lambda) = \alpha c_{yt}^{\alpha-1} c_{ot+1}^{1-\alpha} - \lambda = 0 \tag{54}$$

$$\frac{\partial \mathscr{L}}{\partial c_{ot+1}}(c_{yt}, c_{ot+1}, \lambda) = (1-\alpha) c_{yt}^{\alpha} c_{ot+1}^{-\alpha} - \frac{\lambda}{1+r_{t+1}} = 0 \tag{55}$$

$$\frac{\partial \mathscr{L}}{\partial \lambda}(c_{yt}, c_{ot+1}, \lambda) = w_t - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1}} = 0. \tag{56}$$

If we solve equation (54) for λ we obtain

$$\lambda = \alpha c_{yt}^{\alpha-1} c_{ot+1}^{1-\alpha} \tag{57}$$

and substituting this into equation (55) and simplifying gives

$$c_{ot+1} = \tfrac{1-\alpha}{\alpha} c_{yt}(1+r_{t+1}). \tag{58}$$

And substituting this into equation (56) (and simplifying a bit) gives

$$w_t - c_{yt} - \tfrac{1-\alpha}{\alpha} c_{yt} = 0 \tag{59}$$

which easily simplifies to give

$$c_{ty} = \alpha w_t. \tag{60}$$

And we can obtain the required expression for c_{ot+1} by substituting the solution for c_{yt} into equation (56).

EXERCISE 3.1. Another way of solving this problem is to solve the budget constraint it give c_{ot+1} as a function of c_{yt} (and w_t and r_{t+1}) and to substitute this into the utility function giving a function that depends on c_{yt}, but not on c_{ot+1}, and finding the unconstrained maximum of this function. Solve the problem in this way and confirm that the solution is the same as the one found above.

EXERCISE 3.2. Solve the maximisation problem using the utility function

$$\tilde{u}_t(c_{yt}, c_{ot+1}) = \log c_{yt} + \tfrac{1-\alpha}{\alpha} \log c_{ot+1}$$

first by using the Lagrangian method, and then by substituting the formula for c_{ot+1} into the utility function. Confirm that the solutions for c_{yt} and c_{ot+1} are the same as those found above.

2.5. Special features of our assumptions. The solutions we obtain have a very simple structure. Recall that the solution to the individual's intertemporal optimisation problem was:

$$\begin{aligned} c_{yt} &= \alpha w_t \\ s_{yt} = a_{t+1} &= (1-\alpha) w_t \\ c_{ot+1} &= (1-\alpha)(1+r_{t+1}) w_t \end{aligned}$$

where we have introduced the symbol s_{yt} to indicate the amount the individual saves in period t. Note, in particular that the consumption and saving in period t do not depend on the interest rate in period $t+1$. The simple structure of these solutions depends essentially on the very special structure of our model.

Two aspects in particular are worth mentioning. One may be thought of as part of the *structure* of the model. This is our very special assumption about the individual's labour supply, namely that the individual inelastically supplies one unit of labour in the first period of his life and does not supply any labour in the second period of his life. A more realistic assumption, and one that would allow us to say at least something about employment, would that the individual had some choice about the amount of labour to supply, that is, he had a choice as to whether to supply his time as labour, or to consume it as leisure. Further we might also allow him to supply some labour in the second period.

EXERCISE 3.3. In the appendix to Chapter 2 of Auerbach and Kotlikoff they analyse the individual's problem when he can work in both periods and where he supplies his labour elastically, that is, he can choose to take some of his available time in the form of leisure. They show that in these circumstances the individual's consumption in the first period depends on all the "prices", w_t, w_{t+1}, and r_{t+1}. It is also possible to see that his savings in the first period also depend on w_t, w_{t+1}, and r_{t+1}. Find the individual's savings when young, s_{yt} as a function of w_t, w_{t+1}, and r_{t+1}.

We could think about making the changes to the model separately. Allowing the individual to allocate his time between leisure and labour in any period in which he can work, and allowing him to work in both periods. Can we say which of these changes causes the dependence of $c_{yt}$and s_{yt} on the second period variables, w_{t+1}, and r_{t+1}? Is it that one change causes the dependence and the other does not, or that either change results in dependence, or that it is necessary to make both changes before we get dependence?

If you do not have access to Auerbach and Kotlikoff, there is a summary of the results in the Appendix to Chapter 2 at the end of this section.

The second very special aspect of our assumptions involves the utility function. Recall how we characterised the solution to a utility maximisation problem with Cobb-Douglas utility. Let the sum of the exponents be 1 and the exponent on the consumption of good i be α_i. Suppose that the consumer has wealth w and the price of good i is p_i. The optimal amount to spend on good i is $\alpha_i w$ and so the optimal amount of good i to consume is $\alpha_i w/p_i$. Notice that this is saying that the amount consumed of a good is independent of the prices of the other goods. This result is very special to the Cobb-Douglas utility function.

Lets look a little more closely at what happens to the consumption of good i when the price of good j increases. An increase in the price of good j is a bad thing for the consumer. His budget set moves inwards. Also the relative price of good i and good j changes. As usual, we can decompose the change in the amount of good j consumed into two components, an income effect and a substitution effect. For the Cobb-Douglas utility function all goods are normal goods and so the amount of good i consumed is positively related to income. The effect of a negative change in income is a negative change in the amount of good i consumed. On the other hand if the price of good j increases then the relative price of good i has decreased, that is good i has become relatively cheaper. For the

Cobb Douglas utility function all goods are substitutes and so the substitution effect will be that the consumer will consume less of good j and more of good i.

Thus we have two effects on the consumption of good i from an increase in the price of good j, a negative income effect, and a positive substitution effect. In general we would say that the overall effect was ambiguous. However, for the Cobb-Douglas utility function the income effect is exactly compensated for by the substitution effect and the overall effect is zero. For a more general utility function the consumption of good i would depend on all prices, not just the price of good i.

In our case there are two goods, consumption when young, c_{yt}, and consumption when old, c_{ot+1}; wealth is w_t; the price of consumption when young is 1 and the price of consumption when old is $1/(1+r_{t+1})$. Thus an implication of utility being Cobb-Douglas is that c_{yt} does not depend on r_{t+1}, and thus neither does savings, which is just $w_t - c_{yt}$. For a more general utility function both c_{yt} and savings would depend on r_{t+1}.

Summary of Appendix to Chapter 2 of Auerbach and Kotlikoff. Suppose that both young and old individuals are endowed with one unit of time, so they may both supply labour. Suppose also that rather than supplying their labour elastically they choose to divide their time between work, for which they are remunerated at a rate given by the wage rate of the period, and leisure, which enters their utility function. We now assume that the individual born in period t has a utility function of the form

$$u_t(c_{yt}, c_{ot+1}, l_{yt}, l_{ot+1}) = c_{yt}^{\alpha} c_{ot+1}^{\eta} l_{yt}^{\theta} l_{ot+1}^{1-\alpha-\eta-\theta}, \tag{61}$$

where $0 < \alpha, \eta, \theta, 1-\alpha-\eta-\theta < 1$. This individual's budget constraint becomes

$$c_{yt} + \frac{c_{ot+1}}{1+r_{t+1}} = w_t(1-l_{yt}) + \frac{w_{t+1}(1-l_{ot+1})}{1+r_{t+1}}. \tag{62}$$

The left hand side of the budget constraint id the same as in equation (45) while the right hand side has an additional term for the wage paid to labour in period $t+1$ and in both periods the individual is paid only for the time he does not consume as leisure. We can rewrite this budget constraint so as to emphasise the fact that leisure in each period is a good that the individual chooses to consume

$$c_{yt} + \frac{c_{ot+1}}{1+r_{t+1}} + w_t l_{yt} + \frac{w_{t+1} l_{ot+1}}{1+r_{t+1}} = w_t + \frac{w_{t+1}}{1+r_{t+1}}. \tag{63}$$

If we again apply the fact that with Cobb-Douglas preferences the individual will spend a fraction of his income on a good given by the exponent on that good in the utility function we find

$$c_{yt} = \alpha \left(w_t + \frac{w_{t+1}}{1+r_{t+1}} \right) \tag{64}$$

$$c_{ot+1} = \eta \left(w_t + \frac{w_{t+1}}{1+r_{t+1}} \right)(1+r_{t+1}) \tag{65}$$

$$l_{yt} = \theta \left(w_t + \frac{w_{t+1}}{1+r_{t+1}} \right) \frac{1}{w_t} \tag{66}$$

$$l_{ot+1} = (1-\alpha-\eta-\theta) \left(w_t + \frac{w_{t+1}}{1+r_{t+1}} \right) \frac{1+r_{t+1}}{w_{t+1}} \tag{67}$$

2.6. A first look at aggregates. So far we've look at how some of the individuals in our model will make their decisions, taking as given some of the economy wide prices (namely, the wage rates, the price of labour, and the interest rates, the price of savings or capital services). We now want to examine what this says about the behaviour of some of the important economy wide aggregate variables.

To keep things as simple as possible we shall assume that there is no change in the size of the population from period to period, that is we shall assume that each generation is the same size, and we'll call the number of individuals in each generation N. (Thus at any time the population will consist of $2N$ individuals, N young people and N old people.

Recall that the capital stock at period $t+1$ is obtained from the assets of those individuals born in period t, which we are calling a_{t+1}, the assets that an individual born in period t takes over into period $t+1$. These assets, a_{t+1} are simply the part of the wage w_t that this individual receive in period t that he did not consume, that is, $a_{t+1} = w_t - c_{yt}$, and we have shown that, with the preferences we have assumed, this will be equal to $(1-\alpha)w_t$.

That is, the total capital stock in period $t+1$, which we shall call K_{t+1}, is given by

$$\begin{aligned} K_{t+1} &= Na_{t+1} \\ &= N(w_t - c_{yt}) \\ &= N(1-\alpha)w_t. \end{aligned} \tag{68}$$

Notice that the capital stock in period $t+1$ depends on w_t, the wage rate in period t.

Since we assume that the young people inelastically supply their one available unit of labour, it is even easier to see what the aggregate supply of labour in a period is, namely $N \times 1 = N$. That is, the total supply of labour in period t, which we shall call L_t, is given by

$$L_t = N. \tag{69}$$

These two cases are relatively simple since they involve, in our simple model, aggregating only one generation's behaviour. Sometimes we are interested in aggregating aver all the individuals alive at a particular time. Let us consider (one aspect of) the National Income Accounts, namely the equality of savings and investment. What is savings in period t, which we call S_t? It is simply the difference between total income or total output, which we'll call Y_t and total consumption, the sum of the consumption of the young in period t (the generation born in period t) and the consumption of the old in period t (the generation born in period $t-1$). That is,

$$S_t = Y_t - Nc_{yt} - Nc_{ot}. \tag{70}$$

(At the risk of repeating myself, I draw your attention to the fact that it is c_{ot} in the previous equation, not c_{ot+1}.)

And what is investment in period t? It is simply the difference between the capital stock that comes into period t and the capital stock transferred into the next period, period $t+1$, that is

$$I_t = K_{t+1} - K_t. \tag{71}$$

(So far we don't seem to be doing too well. Equations (70) and (71) don't seem to have too much to do with each other. However, don't despair!)

Total output Y_t may be paid to three sources, the owners of labour are paid wages, the owners of capital are paid interest, and any residual is (supernormal) profits and is paid to the owners of the firms. We'll see in the next section that in this model, due to the constant returns to scale assumption, there are no supernormal profits. Thus we have

$$Y_t = Nw_t + r_t K_t, \tag{72}$$

where Nw_t is the total wages paid to the owners of the labour used in period t and $r_t K_t$ is the total interest paid to the owners of the capital used in period t. If we substitute this into equation (70) we obtain

$$\begin{aligned} S_t &= Nw_t + r_t K_t - Nc_{yt} - Nc_{ot} \\ &= N(w_t - c_{yt}) - (Nc_{ot} - r_t K_t) \\ &= Ns_t - (Nc_{ot} - r_t K_t) \\ &= Na_{t+1} - (Nc_{ot} - r_t K_t) \\ &= K_{t+1} - (Nc_{ot} - r_t K_t). \end{aligned} \tag{73}$$

(And now we seem to be doing a little better. Equation (73) is starting to look a little like equation (71). All that we still have to do is to show that K_t is equal to $Nc_{ot} - r_t K_t$. But this is easy!)

Recall equation (44) and consider it for the period previous to the one we stated it for. Then it will say

$$c_{ot} = a_t(1 + r_t). \tag{74}$$

Thus (remembering that $Na_t = K_t$)

$$\begin{aligned} Nc_{ot} - r_t K_t &= Na_t(1 + r_t) - r_t K_t \\ &= Na_t + r_t Na_t - r_t K_t \\ &= K_t + r_t K_t - r_t K_t \\ &= K_t, \end{aligned} \tag{75}$$

just as we required, and thus we have shown that $S_t = I_t$ as we said we would.

3. Profit maximising by competitive firms

There are a number of ways of thinking of production taking place in this model. Perhaps the simplest is to think of each old person in period t using their assets a themselves and hiring as much labour as they want at the current wage w_t.

This makes for a quite internally satisfactory model, but perhaps we might be concerned that in the actual economy most production is undertaken by firms rather than individuals, and wonder if this would make a difference. In any case let's look first at the case in which each old person owns their own firm and employs the young to work for them. What we mean by the assumption of perfect competition is that the economic agents all act as price takers, that is they take the market prices as given and choose their quantities optimally, given these market prices. Thus the old individuals will choose a quantity of labour l to maximise

$$A_t a_t^{\beta} l^{1-\beta} - w_t l. \tag{76}$$

We solve this problem by differentiating with respect to l and setting the result equal to zero. That is,

$$A_t(1 - \beta) a_t^{\beta} l^{-\beta} - w_t = 0, \tag{77}$$

or

$$\frac{A_t(1 - \beta)}{w_t} a_t^{\beta} = l^{\beta}, \tag{78}$$

or

$$l = \left(\frac{A_t(1 - \beta)}{w_t} \right)^{1/\beta} a_t. \tag{79}$$

Now since there are N old individuals each wanting to employ l units of labour and N young individuals each wanting to supply 1 unit of labour market clearing implies that $l = 1$, and so

$$1 = \left(\frac{A_t(1 - \beta)}{w_t} \right)^{1/\beta} a_t, \tag{80}$$

or

$$w_t = (1 - \beta) A_t a_t^{\beta}. \tag{81}$$

Notice that for *any* values of a_{t+} and w_t there is an optimal (profit maximising) quantity of labour for each old individual to employ. In order to make the total amount of labour demanded equal to the total amount supplied we require that the wage be as in equation (81). Notice also that what this is saying is that in each "firm" there is one old person providing the capital, in the amount a_t and one young person providing the labour in the amount of

1. The total output is $A_t a_t^{\beta}$[7] and of this the young "worker" gets a share $1-\beta$ and the old "capitalist" the remaining share β.

Since each old person is simply using his assets in his own firm there is not, in this version of the model, really any interest rate. However we can find an implicit interest rate by finding how much extra the old person receives for each unit of his assets. The old person invests a_t in the production process and then gets the fraction β of the output $A_t a_t^{\beta}$ and also has a_t to consume after the production is complete. Thus

$$r_t = \frac{\beta A_t a_t^{\beta}}{a_t} = \beta A_t a_t^{\beta-1}. \tag{82}$$

Before proceeding further let us consider a slightly different model of production. Instead of thinking of each old person as using his assets and employing the young as labour we could think of the old as supplying their capital through competitive capital markets and the young supplying their labour through competitive labour markets to some set of competitive firms.

The answers we will get are essentially the same as we obtained above, though the details of the story are a little different. First we need to say who owns the firms and who gets any profits that the firms earn. In a model in which the production technology exhibits constant returns to scale, as it does in our model, competitive firms earn zero profits, so it doesn't really matter who we say gets the profits. For the sake of concreteness we can say that each of the old individuals own an equal share of each of the firms, though and other arrangement of the ownership of the firms will lead to the same outcomes.

The second difference in in how we define equilibrium in our model. The definition is essentially the same. We just need to be a little more explicit and careful about the details. We say that the values of w_t and r_t are equilibrium values if, given these values there are profit maximising quantity choices by the firms such that the total amounts of capital and labour demanded by the firms equals the total amounts that the individuals want to supply.[8] Notice that this definition of equilibrium does not require that there be profit maximising choices at all prices, just that at the equilibrium prices the firms are choosing (one of) the profit maximising combinations of capital and labour.

Since the firm now purchases both capital and labour the firm's problem is to choose amounts k and l to maximise

$$A_t k^{\beta} l^{1-\beta} - r_t k - w_t l. \tag{83}$$

It is, in fact, easy to see that for many prices (that is, values of w_t and r_t) the firms will not have profit maximising choices. Suppose that w_t and r_t are such that there are values $\bar{k}$ and $\bar{l}$ such that the firm's profit is strictly positive, that is

$$A_t \bar{k}^{\beta} \bar{l}^{1-\beta} - r_t \bar{k} - w_t \bar{l} = \Pi > 0. \tag{84}$$

If the firm were to choose values $2\bar{k}$ and $2\bar{l}$ instead then profits would be 2Π, which, since $\Pi > 0$, is greater than Π. Thus whenever the firm is making positive profits it will do better by keeping the same capital labour ratio and increasing the scale of production. This will, in fact, be true for any constant returns to scale production technology.

EXERCISE 3.4. Show that if $\bar{k}$ and $\bar{l}$ are such that the firm's profit is

$$A_t \bar{k}^{\beta} \bar{l}^{1-\beta} - r_t \bar{k} - w_t \bar{l} = \Pi > 0. \tag{85}$$

then if the firm were to choose values $2\bar{k}$ and $2\bar{l}$ instead their profits would be 2Π. Show that this is true not just for the Cobb-Douglas production function, but is true for any constant returns to scale production function.

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[7] Since $l = 1$ and $l^{1-\beta} = 1^{1-\beta} = 1$.

[8] In general, the amounts that the individuals want to supply might depend on w_t and r_t as well, though in our model they do not.

Let the aggregate capital stock be $K_t = Na_t$ and the aggregate labour be $L_t = N$. The prices w_t and r_t will be equilibrium prices if they are equal to the marginal product of labour and the marginal product of capital respectively, when the amounts of labour and capital are L_t and K_t. Recall from Chapter 2 that the marginal product of labour for the Cob-Douglas production function is $(1-\beta)A_tk_t^\beta$ and the marginal product of capital was $\beta A_tk_t^{\beta-1}$. Thus the equilibrium values for w_t and r_t are

$$w_t = (1-\beta)A_tk_t^\beta \tag{86}$$

$$r_t = \beta A_tk_t^{\beta-1}. \tag{87}$$

Now, since the number of old individuals is the same as the number of young individuals the amount of assets of each old person, a_t is the same as the average amount of capital per unit of labour, that is per young person, namely k_t. Thus the equations (86) and (87) we have found for this interpretation of the model are exactly the same as we found for the previous interpretation in equations (81) and (82).

EXERCISE 3.5. Suppose that instead of each old individual using their assets and hiring the young as labour, or firms employing both capital and labour in competitive markets each young person was to use his labour in production and rent capital in competitive capital markets. Formulate the model and find the equilibrium interest rate. Also find an implicit wage rate. Are these the same as the ones we found above?

4. The Transition Equation

We are now in a position to see how the economy changes from one period to the next. We shall work here in per capita terms.

Recall that we found, in equation (86) that

$$w_t = (1-\beta)A_tk_t^\beta$$

and in equation (52) that

$$a_{t+1} = (1-\alpha)w_t.$$

Putting the two of these together, and remembering that $a_{t+1} = k_{t+1}$, gives

$$k_{t+1} = (1-\alpha)(1-\beta)A_tk_t^\beta, \tag{88}$$

an equation giving the capital/labour ratio in period $t+1$ in terms of the capital labour ratio in period t. This equation allows us to describe the path of the economy through time.

We should perhaps comment that the analysis we have already done allows us, once we know k_t in any period to find the other variables. Equations (86) and (87) tell us the wage rate and the interest rate, while equation (60) tells us the consumption of the young in period t, equation (52) how much they save and take over into period $t+1$ as assets, and equation (44) (taken one period back) the consumption of the old.

Let us look a bit more closely at equation (88) giving the capital/labour ratio in period $t+1$ in terms of the capital labour ratio in period t. This function just says that k_{t+1} is a positive constant times k_t raised to a fractional power. When $k_t = 0$ then $k_{t+1} = 0$ so the function passes through the origin. The slope of the function is positive and decreasing. The slope is infinite at zero and becomes very small as k_t becomes large. The graph of such a function is shown in Figure 2.

EXERCISE 3.6. Show that the transition equation (88) giving the capital/labour ratio in period $t+1$ in terms of the capital labour ratio in period t has the properties stated in the text. Namely that when $k_t = 0$ then $k_{t+1} = 0$ so the function passes through the origin, that the slope of the function is positive and decreasing and that the slope is infinite at zero and becomes very small as k_t becomes large. To show most of these properties it is necessary to differentiate the function given on the right hand side of the equation with respect to k_t. The resulting formula gives you the slope of the original graph for that value of k_t.

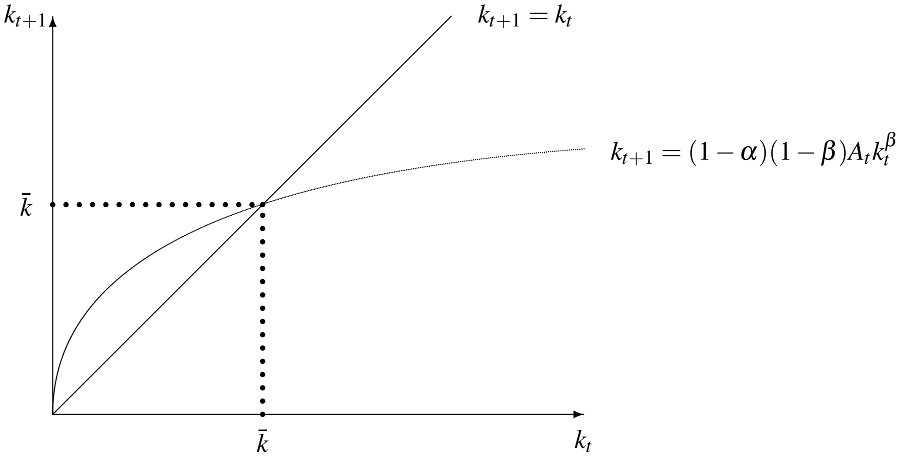

FIGURE 2. The Transition Equation

Now we can argue that the graph of transition function cuts the graph of the line $k_{t+1} = k_t$ (that is, the 45° line) in exactly one place in addition to the origin. We have said that the slope of the function is infinite at 0 so the graph initially goes above the 45° line. We have also said that the slope eventually becomes very small and that it is decreasing so that once it becomes small it stays small. This implies that the graph will eventually cut the 45° line. When it cuts the 45° line it does so from above and so its slope is less than or equal to 1. Since the slope is decreasing in k_t the slope then stays below 1 and so the curve can never cut the 45° line again.

Thus there are exactly two points at which the transition function cuts the 45° line, $(0,0)$ and some other point on the 45° line that we'll label $(\bar{k},\bar{k})$. We can also see from the graph how k_t will evolve through time. If $k_t < \bar{k}$ then the graph of the transition function is above the 45° line and so $k_{t+1} > k_t$, that is the capital labour ratio is increasing. We also see that in this case $k_{t+1} < \bar{k}$, so however many periods we go we do not go above $\bar{k}$. It is also clear that as we go for more and more periods we get very close to $\bar{k}$. On the other hand if $k_t > \bar{k}$ then the graph of the transition function is below the 45° line and so $k_{t+1} < k_t$, that is the capital labour ratio is decreasing. We also see that in this case $k_{t+1} > \bar{k}$, so however many periods we go we do not go below $\bar{k}$. Again, it is clear that as we go for more and more periods we get very close to $\bar{k}$. Finally if $k_t = \bar{k}$ then $k_{t+1} = \bar{k}$ as well and we just stay at $\bar{k}$ forever. For this reason we refer to $\bar{k}$ as the *steady state* level of the capital labour ratio, and describe the economy with this capital labour ratio as being in its steady state.

EXERCISE 3.7. Suppose that $A_t = 10$ in each period, that $\alpha = 0.5$ and $\beta = 0.3$. Find the steady state value of the capital labour ratio, $\bar{k}$. Suppose that the initial capital labour ratio is $k_1 = 1.0$. Find k_2, k_3, k_4, and k_5. Find also w_5 and r_5. Can you say, without explicitly calculating it, if w_6 will be more or less than w_5? What about if r_6 will be more or less than r_5?

CHAPTER 4

Extensions to the Basic Overlapping Generations Model

In this Chapter we examine some extensions to the basic model we developed in the previous chapter. If you understand that model well this material should be quite easy. (On the other hand, if you don't understand that model, the material in this Chapter may well be incomprehensible.) Read the last Section and the Appendix of Chapter 3 and the Section on "The Impact of Government Debt" in Chapter 7 of Auerbach and Kotlikoff.

1. Population Growth

We have made the unrealistic assumption that the number of individuals in each generation is the same. This is clearly unrealistic. For most of human history population has been growing. In recent times is some countries, such as Japan low birth rates have caused the population to begin to fall. The countries of Western Europe also have very low birth rates, though immigration has mitigated the impact of this on their populations.

There are many ways that we could introduce a variable population into our model. We'll consider a model that is only slightly different to the model with a constant population, namely one with a constant population growth rate. That is, rather than assuming that the size of each generation is N we shall assume that the generation born in period t is of size N_t and that the population evolves according to the rule

$$N_{t+1} = (1+n)N_t, \tag{89}$$

that is, in each period there are $1+n$ young workers for every old worker.

How does this change our results? The answer is: not very much. The intertemporal optimisation problem of the individuals is the same and the solutions are again that $c_{yt} = \alpha w_t$ and $a_{t+1} = (1-\alpha)w_t$. Also, given the available capital and labour the firms' profit maximisation problem will be unchanged and so the equations relating the wage rate and the interest rate to the capital labour ratio (equations (86) and (87)) remain unchanged. The only thing that changes is that it is no longer the case that a_{t+1}, the amount of assets that each old person takes into period $t+1$, is equal to k_{t+1} the amount of capital available to each young person in period $t+1$. Now, since there are $1+n$ young worker for each old investor the average amount of capital available for each young worker will be $1/(1+n)$ times the amount of the assets of each old investor. That is,

$$\begin{aligned} k_{t+1} &= \frac{a_{t+1}}{1+n} && (90)\\ &= \frac{(1-\alpha)w_t}{1+n} && (91)\\ &= \frac{(1-\alpha)(1-\beta)Ak_t^{\beta}}{1+n}. && (92)\\ & && (93) \end{aligned}$$

The transition equation has the same qualitative features as the one we found in the previous chapter. In fact the only difference is that the right hand side is divided by $1+n$. Again we can find a steady state value $\bar{k}$ for the capital labour ratio and can trace the path of the other variables, which we can already find in terms of k_t.

2. Exogenous Technological Change

We can add technological change to our model in a similar way to that in which we added population growth. Suppose that the the level of multifactor productivity A changes through time. We indicate this by writing it with a time subscript as A_t. Then our transition equation becomes

$$k_{t+1} = \frac{(1-\alpha)(1-\beta)A_t k_t^\beta}{1+n}. \tag{94}$$

Now, as long as we know the value of A_t in each period (and k_0, the initial value of the capital labour ratio) we can again find the path of the capital labour ratio through time and also find the values of the other variables, which we know as functions of k_t.

Let us consider the special case in which A grows at a constant rate, ν, that is,

$$A_{t+1} = (1+\nu)A_t. \tag{95}$$

Thus

$$\begin{aligned} A_{t+1} &= (1+\nu)A_t \\ &= (1+\nu)^2 A_{t-1} \\ &\vdots \\ &= (1+\nu)^{t+1}A_0. \end{aligned}$$

In this case the capital labour ratio does not converge to a steady state, but continues to grow. In the long run, as t becomes very large the economy does tend towards some form of "steady state," but one in which the capital labour ratio, k_t, grows at a constant rate of $(1+\nu)^{1/(1-\beta)}-1$. That is, for very large t

$$k_{t+1} \approx (1+\nu)^{1/(1-\beta)}k_t. \tag{96}$$

Output per worker, y_t, and the wage rate, w_t will also grow at the approximate rate of $(1+\nu)^{1/(1-\beta)}-1$. The one variable that does not grow in this steady state is the interest rate r_t which converges to a steady state rate $\bar{r}$.

2.1. Derivation of the steady state growth rate of the capital labour ratio. We have claimed that there is a "steady state" in which the capital labour ratio, k_t, grows at the rate $(1+\nu)^{\frac{1}{1-\beta}}-1$, or, somewhat more simply that the growth factor is $(1+\nu)^{\frac{1}{1-\beta}}$. We might ask what we mean by a steady state in which the variable continues to grow, and, if we can find a meaningful answer to that question, how do we see that the growth rate is as we said.

Of course, if k_t is growing forever it doesn't make sense to talk of the steady state value of k_t. What we mean by a steady state in this model is that, at after a large number of periods there are some meaningful parameters that stop changing and converge to a steady state value, and that the other parameters of the model can be expressed as a function, perhaps a function depending on the time period, of these variables.

The idea is to look at the transition equation (94) and restate it in terms that of another variable rather than k_t. We want to choose some function of time that we can multiply k_t by so that this multiple will have a transition function that does not depend on time. We won't go into detail here about how one would go about finding such a function, but simply state the one that works. Thus we define the "artificial" variable q_t as

$$q_t = (1+\nu)^{-\frac{t}{1-\beta}}k_t, \tag{97}$$

and so,

$$k_t = (1+\nu)^{\frac{t}{1-\beta}}q_t. \tag{98}$$

we write equation (97) for period $t+1$ as

$$q_{t+1} = (1+\nu)^{-\frac{t+1}{1-\beta}} k_{t+1}, \tag{99}$$

and recall that our transition function for the variable k_t is

$$k_{t+1} = \frac{(1-\alpha)(1-\beta)(1+\nu)^t A_0 k_t^\beta}{1+n}.$$

Substituting this into equation (99) gives

$$\begin{aligned} q_{t+1} &= \left[\frac{(1-\alpha)(1-\beta)A_0}{1+n}\right](1+\nu)^t \left[(1+\nu)^{\frac{t}{1-\beta}} q_t\right]^\beta (1+\nu)^{-\frac{t+1}{1-\beta}} \\ &= \left[\frac{(1-\alpha)(1-\beta)A_0}{1+n}\right] q_t^\beta (1+\nu)^{\frac{t-t\beta+t\beta-t-1}{1-\beta}} \\ &= \left[\frac{(1-\alpha)(1-\beta)A_0}{(1+n)(1+\nu)^{\frac{1}{1-\beta}}}\right] q_t^\beta. \end{aligned}$$

Thus the transition function is

$$q_{t+1} = \left[\frac{(1-\alpha)(1-\beta)A_0}{(1+n)(1+\nu)^{\frac{1}{1-\beta}}}\right] q_t^\beta. \tag{100}$$

Notice that this transition function does not depend on the time period t.

EXERCISE 4.1. From the transition function (100) find the steady state value of q_t. Consistent with our previous usage lets call this value $\bar{q}$.

Recall now equation (98). If q_t is at its steady state value $\bar{q}$ then this equation becomes

$$k_t = (1+\nu)^{\frac{t}{1-\beta}} \bar{q} = \left[(1+\nu)^{\frac{1}{1-\beta}}\right]^t \bar{q}. \tag{101}$$

Thus when q_t is at its steady state value k_t will grow at the factor $(1+\nu)^{\frac{1}{1-\beta}}$, and when q_t is close to its steady state value k_t will grow at close to this factor.

3. Social Optimality

We shall want to compare different economic policies. So far we have looked at individuals optimising — that is, individuals taking the parameters of the economy as given and asking what is the best they can do. When we look at economic policy we look at things that change some of the parameters and thus have an effect on the welfare of individuals in the economy. If we want to evaluate some economic policy we need some way to aggregate the different effects that this policy may have on different individuals.

One way of aggregating the individuals' welfare is provided by *social welfare function*. A social welfare function is a rule or function that gives, for any profile of utilities of the individuals in the economy, a level of utility or welfare of the whole society. Notice that this implies a greater meaning to the individual utility functions than we usually ascribe to utility functions. If we multiply an individual's utility function by a positive constant we do not change any aspect of the individual's behaviour, but we will, typically change the ranking given by the social welfare function.

Here are some examples of social welfare functions

(1) utilitarian

$$F(u_1, \ldots, u_n) = u_1 + \cdots + u_n$$

(2) weighted utilitarian

$$F(u_1, \ldots, u_n) = \alpha_1 u_1 + \cdots + \alpha_n u_n$$

with each $\alpha_n \geq 0$

(3) Nash, or bargaining function

$$F(u_1,\dots,u_n) = u_1 \times u_2 \times \cdots \times u_n$$

(In this case we require that all the utilities be positive.)

(4) Rawlsian (John Rawls)

$$F(u_1,\dots,u_n) = \min\{u_1,\dots,u_n\}$$

All of these rules have the property that if all the individual utilities increase then the social welfare increases. All but the last have the property that is at least some of the utilities increase and none decrease then the social welfare increases. (We do require that $\alpha_n > 0$ rather than $\alpha_n \geq 0$ in the second example.) Another approach is to only make judgements when this condition is satisfied. That is, we say that one social situation is better than another if and only if all individuals have at least as great a utility and some have strictly more. In this case we say that the situation with greater utility is a Pareto improvement on the other situation, or that the one situation is Pareto superior to the other. If all individuals have greater utility we say the situation is strictly Pareto superior.

4. Government Debt

In an intertemporal setting governments, like individuals, do not have to balance their budgets period by period — and indeed they typically do not do so. Most governments are in debt to the holders of government bonds. Let us call the amount of debt that the government owes at the beginning of period t B_t. Now, during period t the government may either take in more, in the way of taxes and other revenues, than it spends (in which case we say the government's budget is in surplus) or spend more than it takes in (in which case we say the government's budget is in deficit).

We call the difference between expenditures and what the government takes in (most importantly taxes) the budget deficit. Let us denote the deficit during period t as DEF_t. Thus

$$B_{t+1} = B_t + \mathrm{DEF}_t.$$

Let us denote

- G – government purchases of goods and services,
- Z – net taxes (taxes less transfer payments).

Then we can write the government's budget constraint as

$$B_{t+1} = G_t - Z_t + (1+r_t)B_t.$$

We call the term $G_t - Z_t$ the primary deficit for period t. We can rewrite the budget constraint as

$$\frac{B_{t+1}}{R_t} = \frac{G_t - Z_t}{R_t} + B_t,$$

where $R_t = (1+r_t)$. That is, the discounted value of the debt next period is equal to the debt this period plus the discounted value of expenditure this period.

If we take things another step further we can write

$$\frac{B_{t+2}}{R_t R_{t+1}} = \frac{G_{t+1} - Z_{t+1}}{R_t R_{t+1}} + \frac{G_t - Z_t}{R_t} + B_t,$$

and eventually we can write the discounted value of debt far out into the future as

$$\frac{B_T}{R_t R_{t+1} \dots R_T} = B_t + \frac{G_t - Z_t}{R_t} + \frac{G_{t+1} - Z_{t+1}}{R_t R_{t+1}} + \cdots + \frac{G_{T-1} - Z_{T-1}}{R_t R_{T-1}}.$$

Now eventually B_T must grow by less than the interest rate or the interest on government debt would be greater than the entire output of the economy. Thus for T large enough the right hand side of the previous equation is close to zero and so we can write

$$\frac{B_T}{R_t R_{t+1} \dots R_T} = B_t + \frac{G_t - Z_t}{R_t} + \frac{G_{t+1} - Z_{t+1}}{R_t R_{t+1}} + \dots$$

and we can rewrite this as

$$\frac{G_t}{R_t} + \frac{G_{t+1}}{R_t R_{t+1}} + \cdots = -B_t + \frac{Z_t}{R_t} + \frac{Z_{t+1}}{R_t R_{t+1}} + \ldots,$$

where the left hand side gives the present value of purchases and the right hand side is the present values of taxes less the current value of outstanding debt.

4.1. Putting government debt into the model. Since we are going to complicate the model by adding government debt we first simplify it in another way by assuming that consumers consume only when they are old, or, equivalently, that $\alpha = 0$. Thus $c_{yt} = 0$ and $c_{ot+1} = (1 + r_{t+1})w_t$ for all t.

We now add government spending and taxes to our model. Let us suppose that in period t each young person pays z_{yt} and each old person pays z_{ot}.

- z_{yt} – the taxes paid in period t by an individual born at the beginning of period t,
- z_{ot} – the taxes paid in period t by an individual born at the beginning of period $t - 1$.

Thus $Z_t = Nz_{yt} + Nz_{ot}$. We can also write the equation for the consumption of the old in period $t + 1$ (remember that the young don't consume) as

$$c_{ot+1} = (1 + r_{t+1})(w_t - z_{yt}) - z_{ot+1}$$

or

$$c_{ot+1} = (1 + r_{t+1})(w_t - \hat{z}_t),$$

where $\hat{z}_t = z_{yt} + z_{oy+1}/(1 + r_{t+1})$.

Let us recall earlier model without government and add the assumption that $\alpha = 0$. We would then have

$$k_{t+1} = a_{t+1} = w_t = A(1 - \beta)k_t^{\beta}.$$

Once we add the government these are no longer the equalities. The assets that the young person takes over into period $t + 1$ are not w_t but w_t less the taxes that the young person pays in period t. That is

$$\begin{aligned} a_{t+1} &= w_t - z_{yt} \\ &= A(1 - \beta)k_t^{\beta} - z_{yt}. \end{aligned}$$

And the young person does not take over all of his assets in capital goods. Rather he purchases some government bonds. Lets let the amount of government bonds that the young person purchases at the end of period t and holds through period $t + 1$ be b_{t+1}.

Thus

$$k_{t+1} = a_{t+1} - b_{t+1} = w_t - z_{yt} - b_{t+1}.$$

We see that the term that influences the amount of capital that each young person makes available for use in the next period, and so influences the evolution of the economy) is the sum of z_{yt} and b_{t+1}, which we'll call the fiscal policy variable and denote by f_t,

$$f_t = z_{yt} + b_{t+1}.$$

Thus the transition equation becomes

$$k_{t+1} = A(1 - \beta)k_t^{\beta} - f_t.$$

The graph of such a function, with f_t assumed to be constant and equal to $\bar{f}$, is shown in Figure 3.

We see that a positive value of $\bar{f}$ lowers the transition equation and lowers the steady state value of the capital-labour ratio.

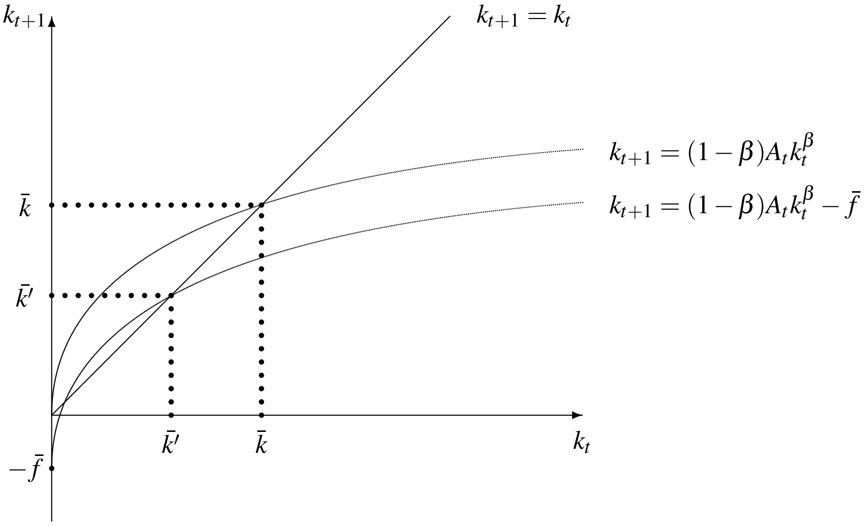

FIGURE 3. The Transition Equation with Government

4.2. Policy examples. We shall now look at a couple of examples of simple government policies. The particular policies that we examine are chosen for their simplicity, not because they are either realistic or desirable.

Let us first consider an economy with a government, but in which the government is currently not doing anything. Suppose that the government decides to, from some period on, purchase an amount $\bar{G}$ of goods and services in each period.

Let us first consider the case in which the government finances this entirely by a tax on the old. Thus

$$z_{ot} = \bar{g} = \frac{\bar{G}}{N}.$$

It is clear that the government satisfies its budget constraint and that government debt is zero in each period. Government purchases $\bar{G} = N\bar{g}$ are equal to net tax payments $Z_t = Nz_{ot}$. Also $f_t = 0$ in each period t and so the the path transition equation does not change, and hence neither does the path of the capital-labour ratio, k_t through time. All that has happened is that the consumption of the old has fallen by exactly the amount of government spending.

Let us now consider a government that seeks to boost the steady state capital-labour ratio by its use of fiscal policy. In order to keep things simple we'll consider the case of a government that makes no purchases of goods and services, so government spending is zero in each period.

Suppose that the government kept debt equal to zero in each period, but transferred resources to the young by making z_{yt} negative. Now, in order to keep government debt equal to zero, with government spending also equal to zero, we must have total taxes equal to zero. Thus, since the generations are of the same size we must have $z_{ot} = -z_{yt} > 0$.

Let us suppose that we are initially in a steady state with $z_{yt} = z_{ot} = b_t = g_t = 0$. Thus the transition equation is

$$k_{t+1} = A(1-\beta)k_t^{\beta}$$

and we find the steady state value $\bar{k}$ by solving

$$\bar{k} = A(1-\beta)\bar{k}^{\beta}$$

or

$$\bar{k}^{1-\beta} = A(1-\beta)$$

or

$$\bar{k} = [A(1-\beta)]^{\frac{1}{1-\beta}}.$$

Also

$$w_t = (1-\beta)Ak_t^{\beta}$$

and so

$$\bar{w} = (1-\beta)A\bar{k}^{\beta} = \bar{k}.$$

And

$$r_t = \beta A k_t^{\beta-1}$$

and so

$$\bar{r} = \beta A\bar{k}^{\beta-1} = \frac{\beta}{1-\beta}.$$

Now lets suppose that in period 0 the government decides to set $z_{yt} = -z$ and $z_{ot} = +z$ (with $z > 0$) and to maintain this policy forever. Let us first look at what the long run effects of this policy will be.

As we have seen, this policy leads to no (additional) government debt, so

$$f_t = z_{yt} + 0 = -z < 0.$$

And the transition equation is

$$k_{t+1} = A(1-\beta)k_t^{\beta} + z.$$

We can find the new steady state value of the capital-labour ratio, which we'll call $\bar{k'}$, by solving

$$\bar{k'} = A(1-\beta)\bar{k'}^{\beta} + z,$$

which, unfortunately, we cannot solve nicely as we could when $z = 0$.

However, we can solve the equation numerically (when we have numbers for A, β, and z) and graphically we can see that $\bar{k'}$ will be higher than $\bar{k}$. In the new steady state we shall have

$$\begin{aligned}\bar{w'} &= (1-\beta)A\bar{k'}^{\beta}\\ &= \bar{k'} - z\\ &> \bar{w}\end{aligned}$$

and

$$\begin{aligned}\bar{r'} &= \beta A\bar{k'}^{\beta-1}\\ &< \bar{r}\end{aligned}$$

EXERCISE 4.2. Examine Figure (4) and make an argument as to why $\bar{w'} < \bar{w}$.

Now, before the taxes, we would have had

$$\begin{aligned}\bar{c}_o &= (1+\bar{r})\bar{w}\\ &= (1+\frac{\beta}{1-\beta})\bar{w}\\ &= \frac{1-\beta+\beta}{1-\beta}\bar{w}\\ &= \frac{1}{1-\beta}\bar{w}.\end{aligned}$$

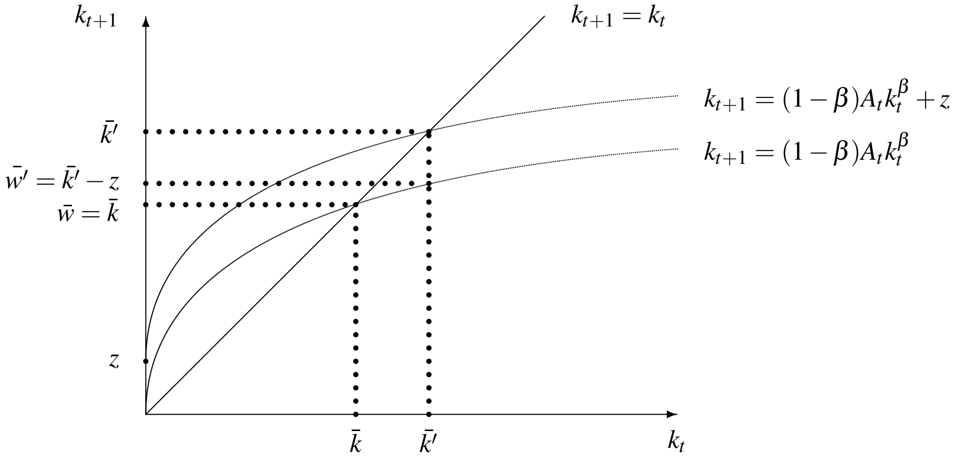

FIGURE 4. The Transition Equation with negative $\bar{f}$

After the taxes we have

$$\begin{aligned}
\bar{c}'_o &= (1+\bar{r}')(\bar{w}'+z) - z \\
&= (1+\bar{r}')(\bar{k}'-z+z) - z \\
&= (1+\bar{r}')\bar{k}' - z \\
&= (1+\beta A\bar{k}'^{\beta-1})\bar{k}' - z \\
&= (\bar{k}'-z) + \beta A\bar{k}'^{\beta}. \qquad (102)
\end{aligned}$$

Now, from the transition equation

$$\bar{k}' = (1-\beta)A\bar{k}'^{\beta} + z$$

or

$$\frac{\bar{k}'-z}{1-\beta} = A\bar{k}'^{\beta} \qquad (103)$$

and substituting this into equation (102) gives

$$\begin{aligned}
\bar{c}'_o &= (\bar{k}'-z) + \frac{\beta}{1-\beta}(\bar{k}'-z) \\
&= (1+\frac{\beta}{1-\beta})(\bar{k}'-z) \\
&= \frac{1-\beta+\beta}{1-\beta}\bar{w}' \\
&= \frac{1}{1-\beta}\bar{w}'
\end{aligned}$$

and we have seen that $\bar{w}' > \bar{w}$ and so $\bar{c}'_o > \bar{c}_o$.

EXERCISE 4.3. We have just seen that a policy of taxing the old and transferring the income to the young will raise the steady state values of the capital-labour ratio, the wage rate, and the consumption of the old (who are the only ones who consume in this model. Does this policy give a Pareto improvement? If so explain why. If not, identify who loses.